Hybrid noiseless subsystems for quantum communication over optical fibres

LETTER TO THE EDITOR

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Abstract
We derive the general structure of noiseless subsystems for optical radiation
contained in a sequence of pulses undergoing collective depolarization in an
optical fibre. This result is used to identify optimal ways to implement quantum
communication over a collectively depolarizing channel, which in general
combine various degrees of freedom, such as polarization and phase, into joint
hybrid schemes for protecting quantum coherence.

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The ability to maintain and transmit coherent quantum superpositions is a prerequisite for
the realization of protocols in quantum information processing. In communication tasks,
such as quantum key distribution [1] and teleportation [2], a natural choice for the physical
carrier of quantum information is optical radiation, often confined to optical fibres in order
to minimize uncontrolled interactions with the environment. Quantum states encoded in low
intensity, narrowband optical fields experience two dominant decoherence mechanisms when
transmitted through optical fibres [1]. The first mechanism is a linear loss of the light amplitude
through residual absorption, scattering and coupling out to non-guided modes. The second
one is a random transformation of the polarization state of the transmitted light due to changes
in fibre birefringence caused by environmental conditions such as temperature fluctuations
and aeolian vibrations. In this paper we address the latter mechanism and analyse ways to
protect quantum coherence against depolarization by using general, multiphoton states of
optical radiation. We assume here that linear losses are polarization independent and that they
can be dealt with in a standard way by postselecting transmissions when no photons have been
lost.

What makes fibre depolarization manageable in quantum communication tasks is that its
temporal variations are relatively slow. In practice, the fibre birefringence remains virtually
constant for long sequences of pulses that can be temporally resolved by the receiver, or even

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for roundtrip transmissions from the receiver to the sender and back. This enables a number of strategies for implementing communication protocols over such collectively depolarizing channels. The first one is to use a degree of freedom that is not affected by depolarization at all, such as the relative phase between consecutive pulses. This idea is best illustrated by time-bin encoding used successfully in a variety of experiments [3]. A practical issue in phase encoding is the necessity to maintain a suitable phase reference, which can be accomplished by active stabilization of distributed interferometers [4], two-way roundtrip communication [5], or one-way autocompensation using photon pairs [6]. The second recently proposed strategy exploits the advances in the theory of noiseless subsystems [7] to construct joint polarization states of several photons that are immune to collective depolarization [8–10]. From the fundamental perspective, such states enable quantum communication without a shared reference frame [11]. Most of the work in this direction is based on results obtained for an ensemble of qubits that undergo an identical symmetrized interaction with the environment. In the optical domain, this model translates into a sequence of pulses containing one photon each and prepared in joint, usually entangled, polarization states.

Sequences of identically polarized pulses employed in phase encoding, or trains of single-photon pulses used to implement noiseless subsystems, form very particular classes of states in the entire Hilbert space describing a set of optical modes travelling in a fibre. The central aim of this paper is to analyse the structure of the complete Hilbert space under collective depolarization and to determine its potential for implementing quantum communication protocols. This approach will provide a general treatment of depolarization in optical fibres that includes a number of previously proposed schemes as special cases. Furthermore, our analysis will reveal the existence of hybrid noiseless subsystems that combine both polarization and multiphoton phase encoding in a non-trivial way. These hybrid subsystems are optimal in terms of required resources, namely the total numbers of temporal slots and photons.

We shall consider general, multiphoton states of optical radiation prepared over N temporal slots that experience identical birefringence. The polarization transformation between a pair of annihilation operators describing two orthogonally polarized modes that occupy one temporal slot is given by an element Ω of the Lie group U(2). The element Ω can be decomposed into a product of the overall phase factor, which we will denote by $e^{-i\alpha(\Omega)}$, and the remaining matrix $e^{i\alpha(\Omega)}\Omega$ with the determinant equal to one which belongs to SU(2). The element Ω induces a unitary transformation $\hat{U}(\Omega)$ in the corresponding two-mode Fock space. For our purposes it is natural to use the Schwinger representation [12] in which the Hilbert space is decomposed into subspaces with a fixed number of photons, which we will denote by l. The polarization transformation in the l-photon subspace is given by $(l + 1)$-dimensional representation of the rotation group for spin $l/2$. Explicitly, the decomposition of $\hat{U}(\Omega)$ in the Schwinger representation has the form:

$$\hat{U}(\Omega) = \bigoplus_{l=0}^{\infty} e^{-il\alpha(\Omega)}\hat{\mathcal{D}}^{l/2}(e^{i\alpha(\Omega)}\Omega) \tag{1}$$

where the matrix $\hat{\mathcal{D}}^{l/2}(e^{i\alpha(\Omega)}\Omega)$ is the corresponding element of the $(l + 1)$-dimensional irreducible representation of SU(2).

The transformation due to depolarization of the entire quantum state $\hat{\varrho}$ of radiation contained in N temporal slots is given by the integral:

$$\hat{\varrho} \mapsto \int_{U(2)} d\Omega [\hat{U}(\Omega)]^{\otimes N} \hat{\varrho} [\hat{U}^{\dagger}(\Omega)]^{\otimes N} \tag{2}$$

where $d\Omega$ is the invariant Haar measure in U(2). We are now interested in identifying the subspaces of the entire Hilbert space that preserve quantum coherence. For this purpose, we

need to expand the N-fold tensor product of the unitary transformation $\hat{U}(\Omega)$ into a direct sum of irreducible representations. If we first divide the N-slot Hilbert space into sectors containing in total exactly L photons, this expansion will in general take the following form:

$$[\hat{U}(\Omega)]^{\otimes N} = \bigoplus_{L=0}^{\infty} e^{-iL\alpha(\Omega)} \bigoplus_{j=(L \bmod 2)/2}^{L/2} K_{NL}^{j} \cdot \hat{\mathcal{D}}^{j}(e^{i\alpha(\Omega)}\Omega). \tag{3}$$

In the above expression, the integers K_{NL}^{j} define how many times the spin-j representation appears in the sector of L photons distributed between N slots, and we have symbolically denoted by a dot the operation of taking a direct sum of K_{NL}^{j} replicas of this representation.

If we now insert the decomposition given in (3) into (2), the invariant integration over Ω will completely remove coherence between subspaces with different L, and also between any two sectors with different j. The latter fact is a result of the canonical property of the products of rotation matrix elements integrated over SU(2). Consequently, the sector of the Hilbert space with given L and j can be treated from the point of view of quantum information processing applications as a noiseless subsystem with dimensionality K_{NL}^{j}. This system can be used for example to implement quantum cryptography protocols based on qudits [13], or any other quantum communication task. Another application for which the above decomposition is relevant is the transmission of classical information [11, 14]. Then the system comprising N slots and containing at most L photons can be used to encode $\sum_{L'=0}^{L} \sum_{j=(L' \bmod 2)/2}^{L'/2} K_{NL'}^{j}$ distinguishable classical messages.

Let us now find and discuss explicit values of multiplicities K_{NL}^{j}. First, we will analyse a restricted case when at most only one photon is allowed to occupy a temporal slot. We can then use the well-known result that for an ensemble of L qubits the allowed values of j are $(L \bmod 2)/2, \ldots, L/2$, and that the spin-$j$ algebra appears $\frac{2j+1}{L+1}\binom{L+1}{L/2-j}$ times in a direct-sum decomposition [15]. When the qubits are realized as polarization states of single photons and we have additional freedom to distribute the photons between N temporal slots, these values are multiplied by the factor $\binom{N}{L}$ which gives the number of configurations to occupy L out of N slots. Thus the dimensionality $\tilde{K}_{NL}^{j}$ of a spin-j noiseless subsystem (labelled with a tilde to distinguish it from the general scenario with no constraints on the number of photons in a slot) is given by

$$\tilde{K}_{NL}^{j} = \binom{N}{L}\frac{2j+1}{L+1}\binom{L+1}{L/2-j}. \tag{4}$$

The dimensionalities of noiseless subsystems for $N, L \leqslant 6$ are collected in table 1. It is instructive to compare these numbers with the case of pure phase encoding, when all the input photons are prepared in an identical polarization. This implies that their joint state of polarization belongs to the completely symmetric subspace with the highest value of j, equal to $L/2$. Therefore phase-encoding schemes are included in table 1 as entries with $j = L/2$, and for clarity they have been underlined. An interesting question is whether the dimension of these subspaces can be enhanced by preparing input photons in non-trivial polarization states. The answer is positive for $N \geqslant 4$ and given by the highest entry in each column marked with an asterisk. These entries correspond to hybrid encodings, where both the phase and the polarization are exploited to protect quantum coherence. The first non-trivial hybrid noiseless subsystem occurs for three photons in four temporal slots, and it is obtained by extending the $j = 1/2$ subspace of three qubits by an additional empty temporal slot. In general, a simple analysis of the sign of the difference $\tilde{K}_{NL}^{j} - \tilde{K}_{NL}^{j-1}$ shows that for given N and L the largest subsystem is obtained for the highest integer (L even) or half-integer (L odd) value of j satisfying $j \leqslant \sqrt{L+2}/2$. Therefore, the largest noiseless subsystem will usually have a hybrid character.

Table 1. Multiplicities $\tilde{K}^j_{NL}$ of spin-j representations for L photons distributed between N slots with at most one photon occupying each slot. Asterisked entries denote hybrid noiseless subsystems with the highest dimensionality in each column.

L	Spin	Number of slots N				
		2	3	4	5	6
1	$j = 1/2$	2	3	4	5	6
2	$j = 1$	1	3	6	10	15
2	$j = 0$	1	3	6	10	15
3	$j = 3/2$		1	4	10	20
3	$j = 1/2$		2	8*	20*	40
4	$j = 2$			1	5	15
4	$j = 1$			3	15	45*
4	$j = 0$			2	10	30
5	$j = 5/2$				1	6
5	$j = 3/2$				4	24
5	$j = 1/2$				5	30
6	$j = 3$					1
6	$j = 2$					5
6	$j = 1$					9
6	$j = 0$					5

It is instructive to identify some earlier proposals in the literature with entries in table 1. For example, the two protocols for quantum key distribution proposed in [8] are based on the cases $N = L = 4$ and $N = L = 3$ with the corresponding values of j equal respectively to 0 and 1/2, and two-dimensional noiseless subsystems. The essential advantage of these protocols is that they do not require a phase reference, in contrast to the most straightforward phase encoding scheme $N = 2$, $L = 1$ that gives a noiseless subsystem of the same dimension. Similarly, the alignment-free test of Bell's inequalities proposed by Cabello [9] uses $N = L = 4$, and table 1 suggests that a similar scheme should be possible also for $N = L = 3$.

Given present advances in the area of scalable linear-optics quantum computing, it is interesting to investigate the asymptotics of hybrid noiseless subsystems for large N. Two relevant figures of merit are the average number of logical qubits per slot that can be encoded in a sequence of N temporal slots, given by the expression:

$$C^{(Q)}_N = \frac{1}{N} \log_2 \left(\max_{L,j} \tilde{K}^j_{NL} \right) \tag{5}$$

and the classical communication capacity of a train of N slots renormalized by the number of slots:

$$C^{(C)}_N = \frac{1}{N} \log_2 \left(\sum_{L=0}^{N} \sum_{j=(L \bmod 2)/2}^{L/2} \tilde{K}^j_{NL} \right). \tag{6}$$

We found numerically by calculating these two quantities up to $N = 10^4$ that they tend to the same constant value $\log_2 3$, with asymptotics following a power law with the approximate value of the exponent $C^{(Q,C)}_N \approx \log_2 3 - \mathrm{O}(N^{-0.84})$, as shown in figure 1. In the same graph, we also depict the average number of photons per slot that are required to achieve these capacities, with the same asymptotic limit of $\langle L \rangle / N = 2/3$. Thus we see that asymptotically the system
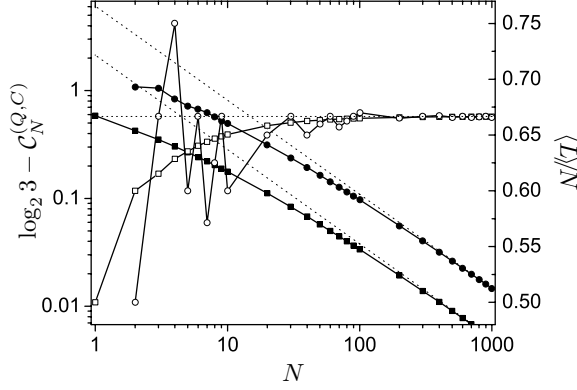
Figure 1. Asymptotics of the capacities $\mathcal{C}_N^{(Q)}$ ($\bullet$) and $\mathcal{C}_N^{(C)}$ ($\blacksquare$) with the corresponding average numbers of photons $\langle L \rangle / N$ for the case of quantum ($\circ$) and classical ($\square$) communication.

behaves as if composed of a train of N noiseless qutrits, each spanned by the vacuum state and two one-photon states with orthogonal polarizations.

Let us now relax the constraint of having at most one photon per temporal slot and analyse the general scenario in which photons can be arbitrarily allocated in the temporal slots. We will derive a closed recursion formula for the multiplicities K_{NL}^j. First, let us simplify the notation by rewriting (3) symbolically in terms of the corresponding algebras of the noise operators:

$$\mathcal{A}_{NL} = \bigoplus_{j=(L \bmod 2)/2}^{L/2} K_{NL}^j \cdot \mathcal{A}^j. \tag{7}$$

Here $\mathcal{A}_{NL}$ is the algebra for L photons in N slots, and the $\mathcal{A}^j$ denote standard irreducible spin-j algebras. The recursion formula will be based on a relation linking the representation multiplicities K_{NL}^j for L photons distributed between N slots to those describing the system with the number of slots reduced by one. When we distribute L photons between N slots, we can put an arbitrary number $L' \leqslant L$ of them into $N-1$ slots we had before, and the remaining $L - L'$ photons into the additional slot. Obviously, the polarization transformation of these $L - L'$ photons in the new slot is governed by the algebra $\mathcal{A}^{(L-L')/2}$. Assuming that the decomposition of the algebra $\mathcal{A}_{N-1,L'}$ for the photons in the remaining $N-1$ slots is known in the form of (7), this gives us the following relation:

$$\mathcal{A}_{NL} = \bigoplus_{L'=0}^{L} \mathcal{A}^{(L-L')/2} \otimes \left(\bigoplus_{j'=(L' \bmod 2)/2}^{L'/2} K_{N-1,L'}^{j'} \cdot \mathcal{A}^{j'} \right). \tag{8}$$

We can now use the standard decomposition of the tensor product of spin algebras:

$$\mathcal{A}^{(L-L')/2} \otimes \mathcal{A}^{j'} = \bigoplus_{j=|(L-L')/2-j'|}^{(L-L')/2+j'} \mathcal{A}^j \tag{9}$$

to obtain an expression for $\mathcal{A}_{NL}$ in the form of a triple direct sum:

$$\mathcal{A}_{NL} = \bigoplus_{L'=0}^{L} \bigoplus_{j'=(L' \bmod 2)/2}^{L'/2} \bigoplus_{j=|(L-L')/2-j'|}^{(L-L')/2+j'} K_{N-1,L'}^{j'} \mathcal{A}^j. \tag{10}$$

We now need to rearrange the summations in order to bring the above expression to the form of (7). This can be done by replacing the first two summation indices L' and j' by a pair of new

Table 2. Multiplicities K_{NL}^{j} for L photons distributed over N slots, including arbitrary multiphoton states in each temporal slot. Underlined entries correspond to pure phase-encoding schemes, whereas asterisks mark optimal hybrid noiseless subsystems for given N and L.

		Number of slots N						
L	Spin	2	3	4	5	6	7	8
2	$j = 1$	3	6	10	15	21	28	36
2	$j = 0$	1	3	6	10	15	21	28
3	$j = 3/2$	4	10	20	35	56	84	120
3	$j = 1/2$	2	8	20	40*	70*	112*	168*
4	$j = 2$	5	15	35	70	126	210	330
4	$j = 1$	3	15	45*	105*	210*	378*	630*
4	$j = 0$	1	6	20	50	105	196	336

integer parameters $\mu = L'/2 + j'$ and $\nu = L'/2 - j'$. Then the two-dimensional grid (L', j') of the summation points with $L' = 0, 1, \ldots, L$ and $j' = (L' \bmod 2)/2, (L' \bmod 2)/2 + 1, \ldots, L'/2$ can be written as $\left(\mu + \nu, \frac{\mu - \nu}{2}\right)$ with the new parameters running through ranges $\nu = 0, 1, \ldots, \lfloor L/2 \rfloor$ and $\mu = \nu, \nu + 1, \ldots, L - \nu$. In the new parametrization, (10) takes the form:

$$\mathcal{A}_{NL} = \bigoplus_{\nu=0}^{\lfloor L/2 \rfloor} \bigoplus_{\mu=\nu}^{L-\nu} \bigoplus_{j=|L/2-\mu|}^{L/2-\nu} K_{N-1,\mu+\nu}^{(\mu-\nu)/2} \mathcal{A}^{j}. \tag{11}$$

If we now want to perform first the summation over j, we need to identify the allowed ranges of μ and ν as a function of j. Considering the limits of the sum over j in (11), we obtain constraints on μ and ν in the form $\nu \leqslant L/2 - j$ and $L/2 - j \leqslant \mu \leqslant L/2 + j$. These constraints are always stronger than the original limits of the sums over μ and ν in (11), and they can replace the latter. Consequently, swapping the order of summations gives the following recursion formula for the multiplicities K_{NL}^{j}:

$$K_{NL}^{j} = \sum_{\nu=0}^{L/2-j} \sum_{\mu=L/2-j}^{L/2+j} K_{N-1,\mu+\nu}^{(\mu-\nu)/2}. \tag{12}$$

In table 2 we calculate explicitly the values of K_{NL}^{j} for $N \leqslant 8$ and $L \leqslant 4$. As before, for each pair of N and L the underlined entry with the highest $j = L/2$ corresponds to pure phase encoding which can be used as a benchmark to indicate whether exploiting the polarization degree of freedom can yield a larger noiseless subsystem. The optimal noiseless subsystems for given N and L in most cases turn out to be hybrid, as shown by asterisked entries in table 2.

In conclusion, we have analysed the structure of the Hilbert space for optical radiation contained in an arbitrary number of temporal modes and undergoing a collective depolarization, which is one of the dominant decoherence mechanisms in optical fibres. We have demonstrated that optimizing communication capacity requires in general a combination of phase and polarization encodings into joint hybrid schemes. Although practical implementation of these schemes may require complex preparation and measurement procedures, present experimental effort towards the realization of scalable linear-optics quantum computing allows for optimism that the necessary techniques will become available. As the hybrid noiseless subsystems utilize relative phases between pulses in a temporal sequence, the communicating parties require in

general a shared phase reference. An interesting question is whether for hybrid subsystems this requirement can be circumvented in a way similar to the autocompensation technique introduced by Walton *et al* [6].

Acknowledgments

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